

THE FIBERWISE INTERSECTION THEORY

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ABSTRACT. We define a bordism invariant for the fiberwise intersection theory. Under some certain conditions, this invariant is an obstruction for the theory.

1. INTRODUCTION

We start with the following assumptions for the intersection theory;

- (i) Let P^p and M^m be smooth manifolds. Suppose $Q^q \subseteq M^m$ is a closed submanifold and $f : P \rightarrow M$ smooth map such that $f \pitchfork Q$ in M .
- (ii) Let $E(f, i_Q) :=$ the homotopy pullback of $[P \xrightarrow{f} M \xleftarrow{i_Q} Q]$. We have a smooth map $f^{-1}(Q) \rightarrow E(f, i_Q)$ and the bundle data determines an element in $\Omega_{p+q-m}^{fr}(E(f, i_Q))$.
- (iii) Let N^{p+q-m} be a submanifold of P and let $N \rightarrow E(f, i_Q)$ be a map which represents the same such an element in $\Omega_{p+q-m}^{fr}(E(f, i_Q))$.

In 1974, Allan Hatcher and Frank Quinn [H-Q] showed in their work that if f is an immersion and assume $m > q + \frac{p}{2} + 1, m > p + \frac{q}{2} + 1$, then we can homotope a map f the the new map g so that $g^{-1}(Q) = N$. We develop their result to the case where f is any smooth map and also weaken the dimension condition as follows; (See [Sun] for more details.)

Theorem 1. (*Classical version*) *Given a smooth map $f : P^p \rightarrow M^m$ and Q^q is a closed submanifold of M . Assume $m > q + \frac{p}{2} + 1$. Then there is a map g homotopic to f such that $g^{-1}(Q) = N$.*

Our proof is using the bundle data to construct the required homotopy step by step. In this paper, we proceed along the same lines as the proof of the classical case to get the result for the fiberwise case which we will describe it in the next section.

2. INTERSECTION THEORY. (FIBERWISE VERSION)

(I) Suppose that E_P^{p+k}, E_Q^{q+k} and E_M^{m+k} are smooth fiber bundles over a compact manifold B^k . Let $f : E_P^{p+k} \rightarrow E_M^{m+k}$ be a bundle map and E_Q be a subbundle of E_M with the inclusion bundle map $i_Q : E_Q \hookrightarrow E_M$.

We have a commutative diagram

$$\begin{array}{ccccc}
 P^p & & M^m & & Q^q \\
 \downarrow & & \downarrow & & \downarrow \\
 E_P & \xrightarrow{f} & E_M & \xleftarrow{i_Q} & E_Q \\
 \searrow \scriptstyle pr_P & & \downarrow \scriptstyle pr_M & & \swarrow \scriptstyle pr_Q \\
 & & B & &
 \end{array} \tag{2.1}$$

where P, Q and M are the fibers of pr_P, pr_Q and pr_M , respectively.

We may assume that $f \pitchfork E_Q$ in E_M (See [Koz]).

The homotopy pullback is

$$E(f, i_Q) := \{(x, \lambda, y) \in E_P \times E_M^I \times E_Q \mid \lambda(0) = f(x), \lambda(1) = y\}.$$

We have a diagram which commutes up to homotopy

$$\begin{array}{ccc}
 E(f, i_Q) & \xrightarrow{\pi_Q} & E_Q \\
 \pi_P \downarrow & & \downarrow i_Q \\
 E_P & \xrightarrow{f} & E_M
 \end{array} \tag{2.2}$$

where π_P and π_Q are the trivial projections, i.e. we have a homotopy

$$E(f, i_Q) \times I \xrightarrow{K} E_M \text{ defined by } K(x, \lambda, y, t) = \lambda(t).$$

We also have a map $c : f^{-1}(E_Q) \rightarrow E(f, i_Q)$ defined by $x \mapsto (x, c_{f(x)}, f(x))$

where $c_{f(x)} = \text{constant path in } E_M \text{ at } f(x)$. Note that $E(f, i_Q)$ and $f^{-1}(E_Q)$ are not necessarily the fiber bundles over B .

Transversality yields a bundle map

$$\begin{array}{ccc}
 \nu_{f^{-1}(E_Q) \subseteq E_P} & \longrightarrow & \nu_{E_Q \subseteq E_M} \\
 \downarrow & & \downarrow \\
 f^{-1}(E_Q) & \xrightarrow{f|_{f^{-1}(E_Q)}} & E_Q
 \end{array} \tag{2.3}$$

Choose an embedding $E_P^{p+k} \subseteq S^{p+k+l}$, for sufficiently large l . Then we get $f^{-1}(E_Q) \xhookrightarrow{i} E_P^{p+k} \subseteq S^{p+k+l}$. So $\nu_{f^{-1}(E_Q) \subseteq S^{p+k+l}} \cong \nu_{f^{-1}(E_Q) \subseteq E_P} \oplus i^* \nu_{E_P \subseteq S^{p+k+l}}$.

The commutative diagram

$$\begin{array}{ccc}
 f^{-1}(E_Q) & \xrightarrow{f|_{f^{-1}(E_Q)}} & E_Q \\
 \searrow c & & \uparrow \pi_Q \\
 & E(f, i_Q) & \\
 \searrow i & \downarrow \pi_P & \\
 & E_P &
 \end{array} \tag{2.4}$$

yields a bundle map

$$\begin{array}{ccc}
 \nu_{f^{-1}(E_Q) \subseteq S^{p+k+l}} & \xrightarrow{\hat{c}} & \pi^*(\nu_{E_Q \subseteq E_M}) \oplus \pi^*(\nu_{E_P \subseteq S^{p+k+l}}) := \xi \\
 \downarrow & & \downarrow \\
 f^{-1}(E_Q) & \xrightarrow{c} & E(f, i_Q)
 \end{array} \tag{2.5}$$

Thus (c, \hat{c}) determines an element $[c, \hat{c}] \in \Omega_{p+q+k-m}^{fr}(E(f, i_Q); \xi)$.

(II) Suppose that $(N \xrightarrow{c_1} E(f, i_Q), \nu_{N \subseteq S^{p+k+l}} \xrightarrow{\hat{c}_1} \xi)$ is another representative of $[c, \hat{c}]$, where $N^{p+q+k-m} \subseteq E_P^{p+k} \subseteq S^{p+k+l}$. This means we have a normal bordism $(W \xrightarrow{c} E(f, i_Q), \nu_W \xrightarrow{\hat{c}} \xi)$ between (c, \hat{c}) and (c_1, \hat{c}_1) , i.e.

- (i) $W^{p+q+k-m+1} \subseteq (S^{p+k+l} \times I)$,
- (ii) $\partial W \subseteq (S^{p+k+l} \times \partial I)$,
- (iii) $W \pitchfork (S^{p+k+l} \times \partial I)$,
- (iv) $W \cap (S^{p+k+l} \times 0) = f^{-1}(E_Q)$ and $W \cap (S^{p+k+l} \times 1) = N$

such that

$$\begin{aligned}
 \mathcal{C}|_{f^{-1}(E_Q)} &= c : f^{-1}(E_Q) \rightarrow E(f, i_Q) \quad , \quad \mathcal{C}|_N = c_1 : N \rightarrow E(f, i_Q) \\
 \hat{\mathcal{C}}|_{\nu_{f^{-1}(E_Q) \subseteq S^{p+k+l}}} &= \hat{c} \quad , \quad \hat{\mathcal{C}}|_{\nu_{N \subseteq S^{p+k+l}}} = \hat{c}_1.
 \end{aligned}$$

Theorem 2. Let M^m and N^n be smooth manifolds and $f : M \rightarrow N$ be a smooth map. If $n > 2m$, then f is homotopic to an embedding $g : M \rightarrow N$.

Proof. See [Muk]. □

Lemma 1. *Let $f : M^m \rightarrow N^n$ be a map between two smooth manifolds. Let A be a closed submanifold of M . Assume that $f|_A$ is an embedding.*

If $n > 2m$, then f is homotopic to an embedding g relative to A .

Proof. Let T be a tubular neighborhood of A in M .

Step I Extend the embedding $f|_A : A \rightarrow N$ to an embedding $f_T : T \rightarrow N$

Let $\nu(A, M)$ be the normal bundle of A in M and $D(\nu)$ denote the disc bundle of ν . Then the tubular neighborhood theorem implies $D(\nu(A, M)) \cong T$.

Claim : For any given an embedding $A \xrightarrow{g} N$ and a vector bundle η over A . Then

$$\begin{aligned} & \left(g \text{ extends to an embedding of } D(\eta) \text{ into } N \right) \\ \iff & \left(\text{There exists a bundle monomorphism } \phi : \eta \rightarrow \nu(g) \right) \end{aligned}$$

where $\nu(g)$ is the normal bundle of A in N via g .

Proof Claim

(\Leftarrow) We have a diagram

$$\begin{array}{ccccc} D(\eta) & \xrightarrow{\phi} & D(\nu(g)) & \hookrightarrow & \text{zero section} \\ & \searrow & \downarrow \text{exp} & & \downarrow \cong \\ & & N & \hookleftarrow & A \end{array} \quad (2.6)$$

where exp is the exponential map.

Note that $\text{exp}(D(\nu(g))) \cong$ tubular neighborhood of A in N via g . Then $\text{exp} \circ \phi : D(\eta) \hookrightarrow N$ is a desired embedding.

(\Rightarrow) Assume there exists an embedding g_T so that the following diagram commutes

$$\begin{array}{ccc} D(\eta) & \xrightarrow{g_T} & N \\ \uparrow i & \nearrow g & \\ A & & \end{array} \quad (2.7)$$

Then $\nu(g) \cong \eta \oplus i^* \nu(g_T)$.

We are in the situation that we have a commutative diagram

$$\begin{array}{ccc} A & & \\ i \downarrow & \searrow f|_A = g & \\ M & \xrightarrow{f} & N \end{array} \quad (2.8)$$

Let $\nu(f) := f^*\tau_N - \tau_M$. Then $i^*(\nu(f)) \oplus \nu(A, M) \stackrel{stable}{\cong} \nu(g)$.

If $n - a > a$, then $i^*(\nu(f)) \oplus \nu(A, M) \cong \nu(g)$, so there exists a bundle monomorphism

$$\begin{array}{ccc} \nu(A, M) & \longrightarrow & \nu(g) \\ & \searrow & \swarrow \\ & A & \end{array} \quad (2.9)$$

Apply Claim when $g = f|_A$ and $\eta = \nu(A, M)$, then we have an extension embedding of g from $D(\nu(A, M)) \cong T \xrightarrow{f_T} N$.

Step 2 We have a map $f|_{M-int(T)} : M \setminus int(T) \rightarrow N$ and $\partial(M \setminus int(T)) = \partial T$.

$b > (c + \frac{a}{2} + 1)$ and Theorem 2 $\Rightarrow f|_{M-int(T)}$ is homotopic to an embedding $g_{M-int(T)}$.

Define $g : M \rightarrow N$ by

$$g(x) = \begin{cases} g_{M-int(T)}(x) & \text{if } x \in M \setminus int(T) \\ g_T(x) & \text{if } x \in T. \end{cases}$$

Then f is homotopic to g relative to A .

□

Theorem 3. Assume $m > q + (\frac{p+k}{2}) + 1$. Then there exists a smooth map over B

$$\Psi : E_P \times I \rightarrow E_M$$

such that $\Psi|_{E_P \times \{0\}} = f$, $\Psi \pitchfork E_Q$ and $\Psi|_{E_P \times \{1\}}^{-1}(E_Q) = N$.

Note that if we let $g = \Psi|_{E_P \times \{1\}}$. Then g is fiber-preserving homotopic to f and $g^{-1}(E_Q) = N$.

Proof. We divide the proof into 3 steps,

Step 1: Goal: Homotope the map $W \xrightarrow{a:=\pi_P \circ \mathcal{C}} E_P$ to an embedding over B .

By assumption, we have

$$W \xrightarrow{\mathcal{C}} E(f, i_Q) \subseteq E_P \times E_M^I \times E_Q.$$

and we also have maps

$$W \xrightarrow{a:=\pi_P \circ \mathcal{C}} E_P \quad , \quad W \xrightarrow{b:=\pi_Q \circ \mathcal{C}} E_Q \quad , \quad W \times I \xrightarrow{H} E_M$$

where $H := K \circ (\mathcal{C} \times id_I)$, so $H|_{W \times 0} = f \circ a$, $H|_{W \times 1} = b$.

Recall that $\partial W = f^{-1}(E_Q) \sqcup N$, $a|_{\partial W}$ is just the inclusion of $f^{-1}(E_Q)$ and N into E_P .

Apply the condition $m > q + \frac{p+k}{2} + 1$ to Lemma 1 , there exists an embedding $\mathcal{A} \simeq a$ (rel ∂W), i.e. we have a commutative diagram

$$\begin{array}{ccc} W \times 1 & & \\ \downarrow & \searrow \mathcal{A} & \\ W \times I & \xrightarrow{L} & E_P \\ \uparrow & \nearrow a & \\ W \times 0 & & \end{array} \tag{2.10}$$

We have a map $W \xrightarrow{b:=\pi_Q \circ \mathcal{C}} E_Q$. By concatenating the homotopy H and $f \circ L$ together, we get a commutative diagram

$$\begin{array}{ccc}
W \times 2 & \xrightarrow{b} & E_Q \\
\downarrow & & \downarrow i_Q \\
W \times [1, 2] & \xrightarrow{V} & E_M \\
\uparrow & & \uparrow f \\
W \times 1 & \xrightarrow{\mathcal{A}} & E_P
\end{array}$$

Thus the following diagram commutes up to homotopy

$$\begin{array}{ccc}
W & \xrightarrow{b} & E_Q \\
\downarrow \mathcal{A} & & \downarrow i_Q \\
E_P & \xrightarrow{f} & E_M
\end{array}$$

Next, we want to modify the homotopy V such that it is fiber preserving with respect to pr_M .

Note that we have a commutative diagram

$$\begin{array}{ccc}
W \times 2 & \xrightarrow{b} & E_Q \\
\downarrow & \nearrow V' & \downarrow pr_Q \\
W \times [1, 2] & \xrightarrow{pr_M \circ V} & B
\end{array} \tag{2.11}$$

We can apply the homotopy lifting property for pr_Q to get a homotopy of b to b' through V' such that the following diagram commute

$$\begin{array}{ccccc}
W \times 2 & \xrightarrow{b' := V'_W \times 1} & E_Q & & \\
\downarrow & & \downarrow i_Q & \searrow pr_Q & \\
W \times [1, 2] & \xrightarrow{V'} & E_Q \subseteq E_M & \xrightarrow{pr_M} & B \\
\uparrow & & \uparrow f & \nearrow pr_P & \\
W \times 1 & \xrightarrow{\mathcal{A}} & E_P & &
\end{array} \tag{2.12}$$

Let $\Psi_W := b' : W \rightarrow E_Q$. Then Ψ_W is a bundle map over B through the lifting V' .

Step 2 Goal: Construct a bundle isomorphism

$$\nu(\mathcal{A}) \oplus \epsilon^1 \cong b'^*(\nu_{E_Q \subseteq E_M})$$

where ϵ^1 is the trivial bundle.

Since $\dim W < \text{rank } \nu(\mathcal{A})$, it is enough to give a stable equivalence between such bundles.

Now, we have

$$W \xrightarrow{\mathcal{A}} E_P \subseteq S^{p+k+l} \implies \nu_{W \subseteq S^{p+k+l}} \cong \nu(\mathcal{A}) \oplus \mathcal{A}^*(\nu_{E_P \subseteq S^{p+k+l}}). \quad (2.13)$$

We also have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{b'} & E_Q \\ & \searrow \mathcal{C} & \downarrow \pi_Q \\ & E(f, i_Q) & \xrightarrow{\pi_Q} E_Q \\ & \downarrow \pi_P & \\ & E_P & \end{array} \quad (2.14)$$

$\mathcal{A} \searrow \quad \quad \quad \downarrow$

$$\mathcal{A} \simeq a = \pi_P \circ \mathcal{C} \implies \mathcal{A}^*(\nu_{E_P \subseteq S^{p+k+l}}) \cong (\pi_P \circ \mathcal{C})^*(\nu_{E_P \subseteq S^{p+k+l}}). \quad (2.15)$$

$$b' \simeq b = \pi_Q \circ \mathcal{C} \implies b'^*(\nu_{E_Q \subseteq E_M}) \cong (\pi_Q \circ \mathcal{C})^*(\nu_{E_Q \subseteq E_M}). \quad (2.16)$$

Thus the bundle map $\hat{\mathcal{C}} : \nu_{W \subseteq S^{p+k+l} \times I} \rightarrow \xi = \pi_P^*(\nu_{E_P \subseteq S^{p+k+l}}) \oplus \pi_Q^*(\nu_{E_Q \subseteq E_M})$ yields the following stable isomorphism

$$\nu_{W \subseteq S^{p+k+l}} \oplus \epsilon^1 \stackrel{\text{stable}}{\cong} (\pi_P \circ \mathcal{C})^*(\nu_{E_P \subseteq S^{p+k+l}}) \oplus (\pi_Q \circ \mathcal{C})^*(\nu_{E_Q \subseteq E_M}). \quad (2.17)$$

Putting (2.13), (2.15) (2.16) and (2.17) together, we get

$$\nu(\mathcal{A}) \oplus (\pi_P \circ \mathcal{C})^*(\nu_{E_P \subseteq S^{p+k+l}}) \oplus \epsilon^1 \stackrel{\text{stable}}{\cong} (\pi_P \circ \mathcal{C})^*(\nu_{E_P \subseteq S^{p+k+l}}) \oplus b'^*(\nu_{E_Q \subseteq E_M}). \quad (2.18)$$

Consequently, we have

$$\nu(\mathcal{A}) \oplus \epsilon^1 \cong b'^*(\nu_{E_Q \subseteq E_M}). \quad (2.19)$$

This implies that we did construct a bundle map

$$\begin{array}{ccc} \nu(\mathcal{A}) \oplus \epsilon^1 & \xrightarrow{\hat{b}'} & \nu_{E_Q \subseteq E_M} \\ \downarrow & & \downarrow \\ W & \xrightarrow{b'} & E_Q \end{array} \quad (2.20)$$

which give us the extension of map b' to the tubular neighborhood of W in E_P . More precisely,

$$\Psi_T : D(\nu(\mathcal{A})) \hookrightarrow D(\nu(\mathcal{A}) \oplus \epsilon^1) \xrightarrow{\hat{b}'} D(\nu_{E_Q \subseteq E_M})$$

where D denotes the disc bundle. Note that $\Psi_T \pitchfork E_Q$ and $\Psi_T(\partial D(\eta_1)) \subseteq E_M \setminus E_Q$.

Since $\nu(\mathcal{A}) \oplus \epsilon^1 \cong b'^*(\nu_{E_Q \subseteq E_M})$, we can find a subbundle η_2 of $b'^*(\nu_{E_Q \subseteq E_M})$ such that $\eta_2 \cong \nu(\mathcal{A})$. For simplicity, let $\eta_1 := \nu(\mathcal{A})$.

Step 3. Goal: Construct the smooth map $\Psi : E_P \times I \rightarrow E_M$ over B .

Recall that we have

$$W \hookrightarrow D(\nu(\mathcal{A})) \simeq D(\nu(\mathcal{A}) \oplus \epsilon^1) \cong D(b'^*(\nu_{E_Q \subseteq E_M})).$$

Then there exists a neighborhood \bar{D} of W in $D(b'^*(\nu_{E_Q \subseteq E_M}))$ such that $\bar{D} \simeq D(b'^*(\nu_{E_Q \subseteq E_M}))$ and $\bar{D} \cong D(\nu(\mathcal{A}))$.

According to (2.12), we have a commutative diagram

$$\begin{array}{ccc} W \times 2 & \xrightarrow{V'|_{W \times 2}} & E_M \times 2 \\ \downarrow & & \downarrow \\ W \times [1, 2] & \xrightarrow{\Psi_1} & E_M \times [1, 2] \\ \uparrow & & \uparrow \\ W \times 1 & \xrightarrow{V'|_{W \times 1}} & E_M \times 1 \end{array} \quad (2.21)$$

where $\Psi_1(w, t) = (V'(w, t), t)$.

Let $\eta_1 = \nu(\mathcal{A} \oplus \epsilon^1)$, $\eta_2 = b'^*(\nu_{E_Q \subseteq E_M})$.

According to (2.20), there exists a bundle η over $W \times I$ such that $\eta|_{W \times i} = \eta_i$ for $i = 1, 2$

Let $D_1 := D(\nu(\mathcal{A}))$, $D_2 := \bar{D}$.

Then $D_i \hookrightarrow D(\eta_i)$ is a homotopy equivalence for $i = 1, 2$ and also $D_1 \cong D_2$.

Since $D_1 \cup W \times [1, 2] \cup D_2 \hookrightarrow D_1 \times [1, 2]$ is a cofibration and a homotopy equivalence, there exist an extension $D_1 \times [1, 2] \xrightarrow{\hat{\Psi}_1} E_M \times [1, 2]$ such that the following diagram commutes

$$\begin{array}{ccc}
D_2 & \xrightarrow{\quad} & D(\nu_{E_Q \subseteq E_M}) \xrightarrow{\text{exp}} E_M \times 2 \\
\downarrow & & \downarrow \\
D_1 \times [1, 2] & \xrightarrow{\hat{\Psi}_1} & E_M \times [1, 2] \\
\uparrow & & \uparrow \\
D_1 & \xrightarrow{f|_{D_1}} & E_M \cong E_M \times 1
\end{array} \tag{2.22}$$

Next we want to construct an embedding $W \xrightarrow{\mathcal{A}'} D_2 \times [2, 3]$ such that the following hold:

- (i) $\mathcal{A}'(W) \cap \{D_2 \times 2\} = f^{-1}(E_Q)$
- (ii) $\mathcal{A}'(W) \cap \{D_2 \times 3\} = N$
- (iii) $\mathcal{A}' \pitchfork D_2 \times \partial[2, 3]$

We start by letting $\alpha : W \rightarrow [2, 3]$ be a smooth map such that $\alpha \pitchfork \partial[2, 3]$, $\alpha^{-1}(2) = f^{-1}(E_Q)$ and $\alpha^{-1}(3) = N$, we also have an inclusion $W \xrightarrow{i_W} D_2$.

Let $\mathcal{A}' := i_W \times \alpha$. Then \mathcal{A}' is such a required map.

By the construction, we have

$$D(\nu(\mathcal{A}')) \cong D_2 \times [2, 3].$$

Let ψ_2 be the composition of the maps

$$W \xrightarrow{\mathcal{A}'} D_2 \times [2, 3] \xrightarrow{\Psi_T|_{D_2 \times id_{[2, 3]}}} M \times [2, 3] \xrightarrow{proj} M$$

Define a map $\Psi_2 := \psi_2 \times \alpha : W \rightarrow E_M \times [2, 3]$.

Using the fact that $D(\nu(\mathcal{A}')) \cong D_2 \times [2, 3]$, then $D_2 \times \partial[2, 3] \cup \mathcal{A}'(W) \hookrightarrow D_2 \times [2, 3]$ is a cofibration and homotopy equivalence. Hence there exist an extension $D_2 \times [2, 3] \xrightarrow{\hat{\Psi}_2} E_M \times [2, 3]$.

Note that for $(x, 3) \in D_2 \times 3$ such that $\hat{\Psi}_2(x, 3) \in E_Q \times 3$, the map $\hat{\Psi}_{2|_{D_2 \times 3}} = \Psi_T$ forces that x has to be in W , so by the definition of Ψ_2 implies $x \in N$. Thus

$$\hat{\Psi}_{2|_{D_2 \times 3}}^{-1}(E_Q) = N.$$

We define a map $\tilde{\Psi} : \{E_P \times I\} \cup \{D_1 \times [1, 2]\} \cup \{D_2 \times [2, 3]\} \rightarrow E_M \times [0, 3]$ by

$$\tilde{\Psi}(p, t) = \begin{cases} (f(p), t) & \text{if } t \in [0, 1] \\ (\hat{\Psi}_1(p), t) & \text{if } t \in [1, 2] \\ (\hat{\Psi}_2(p), t) & \text{if } t \in [2, 3]. \end{cases} \tag{2.23}$$

Then $\tilde{\Psi}$ is well-defined map over B by the construction.

It's not hard to see that $\{E_P \times I\} \cup \{D_1 \times [1, 2]\} \cup \{D_2 \times [2, 3]\}$ is diffeomorphic to $E_P \times I$. Define the map Ψ to be the composition of maps

$$E_P \times I \xrightarrow{\cong} \{E_P \times I\} \cup \{D_1 \times [1, 2]\} \cup \{D_2 \times [2, 3]\} \xrightarrow{\tilde{\Psi}} E_M \times [0, 3] \xrightarrow{proj} E_M$$

where $proj$ is the projection to the first factor.

Thus, we get a map $\Psi : E_P \times I \rightarrow E_M$ over B so that $\Psi|_{E_P \times 0} = f$. By construction, $\Psi \pitchfork E_Q$ and $\Psi|_{E_P \times 1}^{-1}(E_Q) = N$ as required.

□

Corollary 1. *Assume $m > q + (\frac{p+k}{2}) + 1$. Then we can fiber-preserving homotope a map f to the map that its image does not intersect E_Q if and only if $[c, \hat{c}] = 0 \in \Omega_{p+q+k-m}^{fr}(E(f, i_Q); \xi)$.*

3. APPLICATION TO FIXED POINT THEORY.

Let $p : M^{m+k} \rightarrow B^k$ be a smooth fiber bundle with compact fibers and $k > 2$. Assume that B is a closed manifold. Let $f : M \rightarrow M$ be a smooth map over B , i.e. $p \circ f = p$.

The fixed point set of f is

$$Fix(f) := \{x \in M \mid f(x) = x\}.$$

We have a homotopy pull-back diagram

$$\begin{array}{ccc} \mathcal{L}_f M & \xrightarrow{ev_0} & M \\ ev_1 \downarrow & & \downarrow \Delta \\ M & \xrightarrow{\Delta_f} & M \times_B M \end{array} \quad (3.24)$$

where

- (i) $\mathcal{L}_f M := \{\alpha \in M^I \mid f(\alpha(0)) = \alpha(1)\}$,
- (ii) ev_0 and ev_1 are the evaluation map at 0 and 1 respectively,
- (iii) $\Delta :=$ the diagonal map, defined by $x \mapsto (x, x)$,
- (iv) $\Delta_f :=$ the twisted diagonal map, defined by $x \mapsto (x, f(x))$,
- (v) $M \times_B M :=$ the fiber bundle over B with fiber over $b \in B$, given by $F_b \times F_b$ where F_b is the fiber of p over b .

Proposition 1. *There exists a homotopy from f to f_1 such that $\Delta_{f_1} \pitchfork \Delta$.*

The proof relies on the work of Kozniowski [Koz], relating to B -manifolds. Let B be a smooth manifold. A B -manifold is a manifold X together with a locally trivial submersion $p : X \rightarrow B$. A B -map is a smooth fiber-preserving map.

Lemma 2. *Let X and Y be B -manifolds and Z be a B -submanifold of Y . Let $g : X \rightarrow Y$ be a B -map. Then there is a fiber-preserving smooth B -homotopy $H_t : X \rightarrow Y$ such that $H_0 = g$ and $H_1 \pitchfork Z$.*

Proof. See [Cou] for the proof. □

We have a transversal (pullback) square

$$\begin{array}{ccc} \text{Fix}(f_1) & \xrightarrow{i} & M \\ i \downarrow & & \downarrow \Delta \\ M & \xrightarrow{\Delta_{f_1}} & M \times M \end{array} \quad (3.25)$$

where i is the inclusion. Transversality yields that $\nu(i) \cong i^*(\nu(\Delta)) \cong i^*(\tau M)$.

Choose an embedding $M \hookrightarrow S^{m+k}$ for sufficiently large k . Then we have

$$\nu_{\text{Fix}(f_1) \subseteq S^{m+k}} \cong \nu(i) \oplus i^*(\nu_{M \subseteq S^{m+k}}) \cong i^*(\tau M) \oplus i^*(\nu_{M \subseteq S^{p+k}}) \cong \epsilon.$$

We denote this bundle isomorphism by $\nu_{\text{Fix}(f_1) \subseteq S^{m+k}} \xrightarrow{\hat{g}} \epsilon$. We also have a map $\text{Fix}(f_1) \xrightarrow{g} \mathcal{L}_f M$ defined by $x \mapsto c_x$, where c_x is the constant map at x . Thus $L^{\text{bord}}(f) := [\text{Fix}(f_1), g, \hat{g}]$ determines the element in $\Omega_k^{fr}(\mathcal{L}_f M; \epsilon)$.

Applying Theorem 3, We obtain the following corollary;

Corollary 2. *(Converse of fiberwise Lefschetz fixed point theorem)*

Let $f : M^{m+k} \rightarrow M^{m+k}$ be a smooth bundle map over the closed manifold B^k . Assume that $m \geq k + 3$. Then f is fiber homotopic to a fixed point free map if and only if $L^{\text{bord}}(f) = 0 \in \Omega_k^{fr}(\mathcal{L}_f M; \epsilon)$.

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